17 Probability Distribution

Syllabus

Random Variable ● Mean and Variance of a Random Variable ● Probability ● Distribution Probability Mass Function. ● Expected Value and Variance.

A variable is a symbol (A, B, x, y, etc) that can take on any of a specified set of values. When the value of a variable is the outcome of a statistical experiment (i.e. the experiment can have more than one possible outcome), that variable is called a random variable. The system (i.e. a table or an equation) in which the values of a random variable that links the each outcome of statistical experiment with its probability of occurrence is called probability distribution.

Random Variable

Let S be the sample space associated with a given random experiment. Then, a real valued function X which assigns to each event W E S to a unique real number X(w) is called a random variable.

A random variable is a function that associate a unique numerical value with every outcome of an experiment. The value of the random variable will vary from trial to trial as the experiment is repeated. A random variable is usually denoted by the capital letters X, Y, Z, ..., etc.

e.g. A coin is tossed ten times. The random variable X is the number of tails that are noted. X can only take the values 0, 1,2, ... , 10. So, X is a discrete random variable.

There are two types of random variable:

i. Discrete Random Variable

If the range of the real function X : U → R is afinite set or an infinite set of real numbers, it is called a discrete random variable.

e.g. In tossing of two coins S= {HH, HT, TH, IT}, let X denotes number of heads in tossing of two coins, then ,

X (HH) = 2, X (TH) = 1,X (TT) = 0

ii. Continuous Random Variable

If the range of X is an interval (a, b) of R, then X is called a continuous random variable.

e.g. suppose temperature of a city varies between 20°C and 30°C. Then, it can be measured as 25.0003491087^0 C. Thus it can be take any value in the interval (20, 30).

Mean and Variance of a Random Variable

If X is a discrete random variable which assumes values $x_1, x_2, x_3 ...... x_n$ with respective probabilities $P_1, P_2, ...P_n$ then the mean X of X is defined as

$$\bar{X} = p_1(x_1 + \bar{X})^2 \;\; p_2(x_2 + - \bar{X})^2 + p_n(x_n - \bar{X})^2$$

and variance of X is defined as

$$\text{var}(X) = p_1\{x_1 - \bar{X}) + p_n\{x_2 - x)^2$$

$$= \bar{X} = \sum_{i=1}^{n} p_i x_i$$

and variable X is defined as

$$\text{Var}(X) = p_1(x_i - \bar{X})^2 + p_2(x_2 - \bar{X})^2 + ...$$

$$= \sum_{i=1}^{n} p_i (x_i - \bar{X})^2$$

Where, $\bar{X} = \sum_{i=1}^{n} p_i x_i$ is the mean of X

$$\Rightarrow (X) =$$

$$\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right) \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2$$

The square root, of variance gives the standard deviation i.e.

$$\sqrt{\text{var}(x)} = \sqrt{\sigma^2} = \sigma$$

i. The mean of a random variable X is also known as its mathematical expectation or expected value and is denoted by E,{X).

ii. The variance and standard deviation of a random variable are always non-negative.

Important Results on Variance and Standard Deviation

i. Variance

$V(X) = \sigma_x^2 = E(X^2) - \{E\{X)\}^2$

where. $E\{X^2) \sum_{i=1}^{n} x_i^2 P(x_i)$

ii. Standard Deviation

$\sqrt{V(X)} = \sigma_x = \sqrt{E(X^2) - \{(X)\}^2}$

iii. If Y = a X + b. then

(a) E(Y) = E{aX + b) =aE{X) + b

(b) $\sigma_y^2 = V(Y)$ a^2 V(X) = $a^2\sigma_x^2$

(c) $\sigma_y = \sqrt{V(Y)} = |\alpha|\sigma_x$

(iv) If Z = aX^2 + bX + c. then

E (Z)=E{aX^2+ bX+c) = aE{X^2)+bE{X) + c

Example 1

Arandom variable X takes the values 0, 1, 2, 3 and its mean is 1.3.If P (X = 3) = 2 P(X = 1) and P (X = 2) = 0.3. then P(X =O)is

a. 0 b. 0.2

c. 0.3 d. 0.4

Sol: (d) We have, P(X = 0) + P(X = l) + P (X = 2) + P (X = 3) = 1 and 0 × P(X = 0) + 1 × P(x =1) + 2 × P(x = 2) + 3 × P (x = 3) = 1.3

⇒ P(X = 0)+3P (X = l) = 0.7 and 7P (X=1) = 0.7

⇒ P(X = O) + 3P(X=l) = 0.7 and P (X = l) = 0.1

⇒ P(X = O) = 0.4

Probability Distribution

If a random variable X takes values $X_1, X_2 ... X_n$ with respective probabilities P_1, P_2.... P_n then

x	x_i	x_2	x_{3-}	x_a
P(x)	P_i	P_2	$P_{3...}$	P_a

is known as the probability distribution of X. Probability distribution gives the values of the random variable along with the corresponding probabilities.

It satisfies the following conditions.

i. $0 \le P(x_i) \le 1$ ii. $\sum P(x_i) = 1$ P{x_i)=l

e.g. Probability distribution when two coins are tossed.

Let X denotes the number of heads occurred. then P (X = 0) = Probability of occurrence of zero head

$= P(TT) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{4}$

P(X = 1) = Probability of occurrence of one head

$= P(HT) + P(TH) = \frac{1}{2}.\frac{1}{2} + \frac{1}{2}.\frac{1}{2} = \frac{1}{2}$

P(X =2) = Probability of occurrence of two heads

$= P(HH) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$

Thus, the probability distribution when two coins are tossed is as given below:

x	0	1	2
P(x)	1/4	1/2	1/4

Probability Mass Function

A probability function that specifies the probability that any single value of discrete random variable will occur. is called a probability mass function (abbreviated as pmf). If f{x) is the probability mass function of the random variable X. then f{x) = P(X = x) has the following properties :

i. f{x) ≥ 0 for all values of X

ii. Σ f{x)=1

Example 2

If the function P (x = x) = px, x =1,2,3,4,5 else 0, is a probability mass function (pmf), then p is equal to

a. $\frac{1}{14}$ b. $\frac{1}{15}$

c. $\frac{1}{16}$ d. None of these

Sol (b) ∵ p is a pmf.

$\therefore \sum_{x=1}^{5} p(X = x) = 1$

$\therefore$ p(X = 1) + p (X = 2) + ... + p(X = 5) = 1

$\therefore$ p(1)+ p(2) + ...+ p(5) = 1

$\therefore$ 15p =1

$\therefore$ p = $\frac{1}{15}$

Cumulative Mass Function

If X is a discrete random variable with pmf f(x~ its cumulative mass function (abbreviated as emf) specifies the probability that an observed value of X will be no greater than x. That is, if f(x) is a emf and f(x) is a pmf, then F(x) $\neq$ P(X $\leq$ x) = f(x $\leq$ x),

Continuous Probability Function

A probability function for a continuous random variable is called a continuous probability function, since the domain of the function is continuous.

Probability Density Function

For a continuous random variable, the corresponding function f(x) is called a probability density function (abbreviated as pdf). Unlike a prof, a pdf does not specify probabilities for specific individual values of the random variable.

Cumulative Density Function

Corresponding to the cumulative mass function of a discrete random variable, the cumulative density function (abbreviated as cdf) of a continuous random variable specifies the probability that an observed value of X will be no greater than x.

- **Example 3**

X is a continuous random variable with probability density function

$$f(x) = \frac{x^2}{8}; 0 \leq x \leq 1$$

'Then, the value of p $\left(\frac{1}{5}\right) \leq X \leq \frac{1}{2}$ is

a. $\frac{0.117}{24}$ b. $\frac{0.112}{24}$

c. $\frac{0.113}{36}$ d. $\frac{0.112}{36}$

Sol (a) P = $\frac{1}{5} \leq X \leq \frac{1}{2}$

a. $\frac{0.117}{24}$ b. $\frac{0.112}{24}$

c. $\frac{0.113}{36}$ d. $\frac{0.112}{36}$

Expected Value and Variance

The probability distribution provides a model for the theoretical frequency distribution of a random variable and hence must possess a mean, ariance and other descriptive measures associated with the theoretical population which it represents. The average value of a random variable is called the expected value of the random variable.

Let X be a discrete random variable with probability distribution P(X), then the expected value E(X) is given by E(X) = $\Sigma X . P(X)$

where, the elements are summed over all values of the random variable X.

In other words, if a discrete random variable X has possible values $x_1, x_2, ..., x_n$' with corresponding probabilities $P(x_2\ P(x_n)' \cdots 'P(x_n)$ then the expected value E(X) is defined as

Thus, the expected value of random variable X is merely the arithmetic mean which may be denoted by

$$\text{Var}(X) = \sigma^2 = E[X - E(X)^2]$$

$$= \Sigma[X - E(X)]^2 P(X)$$

$$= E(X^2) - (E(X)]^2$$

The standard deviation, σ is the square root of variance.

Properties of Expected Value and Variance

There are several important properties of expected value and variance which allow computational shortcuts:

i. The expected value of a constant c is equal to the constant.

ii. The expected value of the product of a constant c and a random variable X is equal to the constant times the expected value of the random variable.

$$E(c) = eE (c)$$

iii. The expected value of the sum of a random variable X and a constant e is the sum of the expected value of the random variable and the constant.

$$E (X + C) = E(X) + e$$

iv. The expected value of the product of two independent random variables is equal to the product of thier individual expected values.

$$E(XY) = E(X) E(Y)$$

v. The expected value of the sum of the two independent random variables is equal to the sum of their individual expected values.

$$E(X + Y) = E(X) + E(Y)$$

vi. The variance of the product of a constant and a random variable X is equal to the constant squared times the variance of the random variable X.

$$Var (cX) = c^2 Var (X)$$

vii. The variance of the sum of two independent random variables equals the sum of their individual variances. Also, the variance of the difference of two independent random variables equals the sum of their individual variances.

$$Var (X – X) = c^2 Var (X) + Var (Y)$$

Example 4

A random variable X has the following probability distribution

x	–2	–1	0	1	2	3
P(x)	0.1	t	0.2	2t	0.3	t

Then, the expected value is

a. 0.6 b. 0.S

c. 0.7 d. 0.8

Sol (d) The sum of all the probabilities in a probability distribution is always unity.

$\therefore$ 0.1 t + 0.2 + 2t + 0.3 + t = 1

$\Rightarrow$ 0.6 + 4t =1

$\Rightarrow$ 4t = 0.4

$\Rightarrow$ t = 0.1

E(X) = (– 2)(0.1)+ (–1)(0.1) + 0 (02)

+ 1(2×0.1) + 2(0.3) + 3(0.1) = 0.8

Exercise - 1
(Topical Problems)

Random Variable and Its Distribution

1. A random variable X has the following probability distribution. Then, the value of

x	0	1	2	3	4	5	6	7...
P(x)	0	k	2k	3k	2k	k^2	$2k^2$	$7k^2+k$

(i) k (ii) P(X < 3)
(iii) P(X > 6) (iv) P(0 < X < 3)
are respectively

a. $\frac{1}{10},\frac{3}{10},\frac{17}{100}$ and $\frac{3}{10}$ b. $\frac{1}{10},\frac{3}{10},\frac{3}{10}$ and $\frac{17}{100}$

c. $\frac{17}{100},\frac{1}{10},\frac{3}{10}$ and $\frac{3}{10}$ d. None of these

2. Anil's company estimates the net profit on a new product, it is launching, to be Rs. 3,000,000 during the first year if it is successful, Rs. 1,000,000 if it is moderately successful and a loss of Rs. 1,000,000 if it is 'unsuccessful'. The company assigns the following probabilities to first year prospects for the product, successful: 0.15, moderately successful 0.25 and unsuccessful 0.60. Then, the expected value and standard deviation of first year net profit for the product (in million) is

a. 1.48 b. 12.40
c. 13.8 d. None of these

3. Following is the probability density function $f(x) = pxe^{-4x2}$, $0 \le x \le \infty$. Then,the value of p is

a. 8 b. 5
c. 7 d. –1

4. Let X denotes the sum of the numbers obtained when two fair dice are rolled. The variance and standard deviation of X are

a. $\frac{31}{6}$ and $\sqrt{\frac{31}{6}}$ b. $\frac{35}{6}$ and $\sqrt{\frac{35}{6}}$

c. $\frac{17}{6}$ and $\sqrt{\frac{17}{6}}$ d. None of these

5. If the probability density function of a continuous random variable X is

$f(x) = \frac{3+2x}{18}$; $2 \le x \le 4$

= 0; 2 < 2 or x > 4

Then, the mathematical expectation of X is

a. $\frac{83}{27}$ b. $\frac{27}{83}$

c. $\frac{87}{23}$ d. $\frac{38}{72}$

6. A class has 15 students whose ages are 14, 17, 15, 14,21,17,19,20,16,18,20,17,16,19 and 20 yr. One student is selected in such a manner that each has the same chance of being of chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Mean, variance and standard deviation (SO) of X, are respectively

a. 17.53,4.8 and 2.19
b. 2.19,4.8 and 17.53
c. 17.53, 2.19 and 4.8
d. None of these

7. In a trial, the probability of success is twice the probability of failure. In six trials, the probability of atleast four successes will be

a. $\frac{496}{729}$ b. $\frac{400}{729}$

c. $\frac{500}{729}$ d. $\frac{700}{729}$

8. A discrete random variable X has the following probability distribution

X	1	2	3	4	5	6	7
P(X)	C	2C	2C	3C	C^2	$2C^2$	$7C^2+C$

The value of C and the mean of the distribution are

a. $\frac{1}{10}$ and 3.66 b. $\frac{1}{20}$ and 2.66

c. $\frac{1}{15}$ and 1.33 d. None of these

9. If pdf of a crv X is

$f(x) = ae^{-ax}$; $x \ge 0, a > 0$

If P (0 < X < K) = 0.5, then K is equal to

a. $\frac{1}{8}$ b. $\frac{1}{a}\log 2$

c. $\frac{1}{2}\log 2$ d. $\frac{1}{a}\log a$

10. A box contains 100 bulbs, out of which 10 are defective. A sample of 5 bulbs is drawn. The probability that none is defective, is

a. $\frac{9}{10}$ b. $\left(\frac{1}{10}\right)^5$

c. $\left(\frac{9}{10}\right)^5$ d. $\left(\frac{1}{2}\right)^5$

11. If the pdf of a crv X is $f(x) = \frac{x}{8}$, $0 < x < 4$

= 0, Elsewhere

Then, P (X < 1)and P (X ≥ 2) are

a. $\frac{1}{16}, \frac{3}{4}$ b. $\frac{1}{4}, \frac{3}{8}$

c. $\frac{5}{8}, \frac{7}{16}$ d. None of these

12. In a dice game, a player pays a stake of Rs.1 for each throw of a dice. She receives Rs 5, if the dice shows Rs. 3, Rs. 2, if the dice shows a 1 or 6 and nothing otherwise. What is the player's expected profit per throw over a long series of throws?

a. 0.50 b. 0.20

c. 0.70 d. 0.90

13. For a random variable X, E(X) = 3 and E(X2) = 11. Then, variance of X is

a. 8 b. 5

c. 2 d. 1

14. The probability distribution of a random variable X is given as

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
P(X)	p	2p	3p	4p	5p	7p	8p	9p	10p	11p	12p

Then, the value of p is

a. $\frac{1}{72}$ b. $\frac{3}{73}$

c. $\frac{5}{72}$ d. $\frac{1}{74}$ e. $\frac{1}{73}$

15. If the random variable X takes the values $x_1', x_2, x_3, \ldots x_{10}$ with probabilities P (x = xi) = ki, then the value of k is equal to

a. $\frac{1}{10}$ b. $\frac{1}{4}$

c. $\frac{1}{55}$ d. $\frac{7}{12}$ e. $\frac{3}{4}$

16. If m and σ^2 are the mean and variance of the random variable X, whose distribution is given by

X	0	1	2	3
P(x)	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

Then

a. $m = \sigma^2 = 2$ b. $m = 1, \sigma^2 = 2$

c. $m = \sigma^2 = 1$ d. $m = 2, \sigma^2 = 2$

17. A random variable X has the probability distribution

X	1	2	3	4	5	6	7	8
P(x)	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

for the events E = {X is a prime number} and F = {X < 4}, then P (E U F) is MPPET2009

a. 0.77 b. 0.87

c. 0.35 d. 0.50

18. The distribution of a random variable X is given below

X	–2	–1	0	1	2	3
P(X)	$\frac{1}{10}$	k	$\frac{1}{5}$	24	$\frac{3}{10}$	k

The value of k is

a. $\frac{1}{10}$ b. $\frac{2}{10}$

c. $\frac{3}{10}$ d. $\frac{7}{10}$

19. A random variable X has the following probability distribution

X	1	2	3	4
P(X)	k	2	3k	4k

Then, the mean of X is

a. 3 b. 1
c. 4 d. 2

20. If the range of a random variable X is $\{0, 1, 2, 3, 4...\}$ and $P(X = k) = \frac{(K+1)a}{3^k}$) a for $k \geq 0$ then a is equal to

a. $\frac{2}{3}$ b. $\frac{4}{9}$
c. $\frac{8}{27}$ d. $\frac{16}{81}$

21. Two cards are drawn successively with replacement from a well shuffled deck of 52 cards, then the mean of the number of aces is

a. $\frac{1}{13}$ b. $\frac{3}{13}$
c. $\frac{2}{13}$ d. None of these

22. A random variable X takes values 0, 1, 2, 3, ... with probability $P(X = x) = k(x+1)\left(\frac{1}{5}\right)^x$, where k is constant, then $P(X = 0)$ is Kerala

a. $\frac{7}{25}$ b. $\frac{18}{25}$
c. $\frac{13}{25}$ d. $\frac{19}{25}$
e. $\frac{16}{25}$

23. If the mean and variance of a Binomial variate X are 8 and 4 respectively, then $P(X < 3)$ equals

a. $\frac{265}{2^{15}}$ b. $\frac{137}{2^{14}}$
c. $\frac{137}{2^{16}}$ d. $\frac{265}{2^{16}}$

Exercise - 2
(Topical Problems)

24. If a random variable X has the following probability distribution values of X.

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^+ + k$

Then, $P(X \geq 6)$ is equal to

a. $\frac{19}{100}$ b. $\frac{81}{100}$

c. $\frac{9}{100}$ d. $\frac{91}{100}$

2. X is a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{x}{6} + k; & 0 \leq x \leq 3 \\ 0; & \text{otherwise} \end{cases}$$

The value of k is equal to

a. $\frac{1}{12}$ b. $\frac{1}{3}$

c. $\frac{1}{4}$ d. $\frac{1}{6}$

3. A random variable X has the probability distribution given below

X	1	2	3	4	5
$P(X = x)$	K	2K	3K	2K	K

Its variance is

4. If the pdf of a crv X is
$f(x) = k.e^{-\theta x}$, $8 > 0, 0 \leq x \leq \infty$
$= 0, -\infty < x < 0$, then k is equal to

a. 1 b. $\frac{\theta}{2}$

c. θ d. 2θ

5. A random variable X takes values 1, 2, 3, 4 with probabilities $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$ respec t'IveIy, then iIts mean and variance are equal to

a. $\frac{5}{2}, \frac{11}{12}$ b. $\frac{5}{2}, \frac{11}{16}$

c. $\frac{5}{2}, \frac{11}{16}$ d. $\frac{5}{3}, \frac{11}{12}$

6. A function is defined as

$$f(x) = \begin{cases} 0, & \text{for } x < 2 \\ \frac{2x+3}{18} & \text{for } 2 \leq x \leq 4 \\ 0, & \text{for } x > 4 \end{cases}$$

Then, $P(2 < X < 3)$is

a. $\frac{5}{9}$ b. $\frac{4}{9}$

c. $\frac{7}{9}$ d. $\frac{2}{9}$

7. A random variable has the following probability distribution

X	0	1	2	3	4	5	6	7
P(x)	0	2p	2p	3p	p^2	$2p^2$	$7p^2$	2p

The value of P is

a. 1/10 b. – 1

c. –1/10 d. None of these

8. A random variable X has the following probability distribution.

$X = x_1$	1	2	3	4
$P(X = x_1)$	0.1	0.2	0.3	0.4

The mean and standard deviation of X are respectively

a. 2 and 3 b. 3 and 1

c. 3 and $\sqrt{2}$ d. 2 and 1

9. If X is a random variable with distribution given below

X	0	1	2	3
$P(X = x)$	k	3k	3k	k

The value of k and its variance are

a. 1/8, 22/27 b. 1/8, 23/27

c. 1/8, 24/27 d. 1/8, 3/4

10. If a crv X has probability density function (pdf)
$f(x) = ax, 0 \le x \le 1$
$= a, 1 \le x \le 2$
$= 3a \le ax, 2 \le x \le 3$
$= 0$ otherwise
Then, a is equal to

a. 1 b. $\frac{1}{3}$
c. $\frac{1}{2}$ d. $\frac{1}{5}$

11. For a random variable X, if E(X) = 5 and V(X) = 6, then E (X^2) is equal to
a. 19 b. 31
c. 61 d. 11

12. Probability distribution of a drv X is

X	0	1	2	3	4
P(X = x)	k	2k	4k	2k	k

If a = P ($X \ge 2$) and b = P (X < 3), then
a. a < b b. a > b
x. a = b d. None of these

13. The pdf of a Cry X is $f(x) = \frac{k}{\sqrt{x}}, 0 < x < 4$
Then, P ($X \ge 1$) is equal to
a. 0.2 b. 0.3
c. 0.4 d. 0.5

14. The random variable X has the following probability distribution.

x	–3	–1	0	1	3
P(X = x)	0.05	0.45	0.20	0.25	0.05

Then, its mean is
a. –0.2 b. 0.2
c. – 0.4 d. 0.4

15. The probability distribution of the random variable X is given by

x	1	2	3	4
P(X = x)	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{4}$

Then, the value of V(X) is equal to
a. 02 b. 1
c. $\frac{1}{2}$ d. $\frac{2}{5}$

16. f the pdf of a crv X is
$f(x) = 3(1 - 2x^2), 0 < x < 1$
$= 0, x \le 0$ or $x \ge 1$
Then, the cdf of X is F (x) is equal to
a. $2x - 3x^2$ b. $3x - 4x^3$
x. $3x - 2x^3$ d. None of these

17. If the probability distribution of a random variable X is as given below

X = x	–2	–1	0	1	2	3
P(X = x)	$\frac{1}{10}$	k	$\frac{1}{5}$	2k	$\frac{3}{10}$	k

Then, the value of k is
a. $\frac{1}{10}$ b. $\frac{2}{10}$
c. $\frac{3}{10}$ d. $\frac{7}{10}$

18. If random variable X has the following probability distribution.

X	0	1	2	3	4	5	6	7	8
P(X = x)	a	3a	5a	7a	9a	11a	13a	15a	17a

a. $-\frac{7}{4}$ and $\frac{1}{2}$ b. $-\frac{1}{16}$ and $\frac{5}{16}$
c. $-\frac{7}{4}$ and $\frac{5}{16}$ d. $-\frac{1}{16}$ and $\frac{5}{4}$

19. A random variable X takes values –1, 0, 1,2 with Probabilities $\frac{1+3p}{4}, \frac{1-p}{4}, \frac{1+2p}{4}, \frac{1-4p}{4}$
respectively, where p varies over R.
Then, the minimum and maximum values of the mean of X are respectively
a. $-\frac{7}{4}$ and $\frac{1}{2}$ b. $-\frac{1}{16}$ and $\frac{5}{16}$
c. $-\frac{7}{4}$ and $\frac{5}{16}$ d. $-\frac{1}{16}$ and $\frac{5}{4}$

20. If the mean and variance of a random variable X having a Binomial distribution are 4 and 2 respectively, then find the value of P(X = 1).
a. 1/4 b. 1/16
c. 1/8 d. 1/32

21. A random variable X is defined by

$$X = \begin{cases} 3 \text{ with probability } = \dfrac{1}{3} \\ 4 \text{ with probability } \dfrac{1}{3} \\ 12 \text{ with probability} = \dfrac{5}{12} \end{cases}$$

X = 4 with probability =24

Then, E (X) is

a. 6 b. 7

c. 5 d. 8

22. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing is

a. $\frac{17}{3^5}$ b. $\frac{13}{3^5}$

c. $\frac{11}{3^5}$ d. $\frac{10}{3^5}$

Answers

Exercise 1

1. (a) 2. (a) 3. (a) 4. (b) 5. (a) 6. (a) 7. (a) 8. (a) 9. (b) 10. (c)

11. (a) 12. (a) 13. (c) 14. (a) 15. (c) 16. (c) 17. (a) 18. (a) 19. (a) 20. (b)

21. (c) 22. (e) 23. (c)

Exercise 2

1. (a) 2. (a) 3. (b) 4. (c) 5. (a) 6. (b) 7. (a) 8. (b) 9. (d) 10. (c)

11. (b) 12. (c) 13. (d) 14. (a) 15', '(9) 16. (c) 17. (a) 18. (d) 19. (d) 20. (d)

21. (b) 22. (c)

Solutions

Exercise 1

○ Random Variable and Its Distribution

1. (*a*) (i) It is known that the sum of a probability distribution of random variable is one i.e. $\Sigma P(X) = 1$, therefore

$P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) = 1$

$\Rightarrow \quad 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow \quad 10k^2 + 9k - 1 = 0$

$\Rightarrow \quad 10k^2 + 10k - k - 1 = 0$

$\Rightarrow \quad 10k(k+1) - 1(k+1) = 0$

$\Rightarrow \quad (k+1)(10k-1) = 0$

$\Rightarrow \quad k + 1 = 0$ or $10k - 1 = 0$

$\Rightarrow \quad k = -1$ or $k = \frac{1}{10}$

$k = -1$ is not possible as the probability of an event is never negative.

$\therefore \quad k = \frac{1}{10}$

(ii) $P(X < 3) = P(0) + P(1) + P(2) = 0 + k + 2k$

$= 0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \qquad \left[\text{put } k = \frac{1}{10}\right]$

(iii) $P(X > 6) = P(7) = 7k^2 + k$

$= \frac{7}{100} + \frac{1}{10} = \frac{17}{100} \qquad \left[\text{put } k = \frac{1}{10}\right]$

(iv) $P(0 < X < 3) = P(1) + P(2)$

$= k + 2k = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \qquad \left[\text{put } k = \frac{1}{10}\right]$

2. (*a*) The probability distribution of net profit (X) of the new product in the first year is given to be

Profit (in ₹ million) *X*	3	1	-1
Probability *P*(*X*)	0.15	0.25	0.60

Therefore, expected value of profit is given by

$E(X) = \Sigma X\,P(X) = 3 \times 0.15 + 1 \times 0.25 - 1 \times 0.60$

$= ₹\ 0.10$ million $= ₹\ 1,00,000$

$E(X^2) = \Sigma x^2 P(X)$

$= 9 \times 0.15 + 1 \times 0.25 + 1 \times 0.6 = ₹\ 2.20$ million

$\text{Var}(X) = E(X^2) - \{E(X)\}^2$

$= 2.20 - (0.10)^2 = 2.19$

$\text{SD} = \sigma = \sqrt{219}$

$= ₹\ 1.48$ million

3. (*a*) $\int_0^\infty f(x)dx = 1$

$\Rightarrow \quad \int_0^\infty pxe^{-4x^2}.dx = 1 \quad \Rightarrow \quad \frac{p}{2}\left[\frac{e^{-4x^2}}{-4}\right]_0^\infty = 1$

$\Rightarrow \quad \frac{-p}{8}\left[\frac{1}{e^{4x^2}}\right]_0^\infty = 1 \quad \Rightarrow \quad \frac{-p}{8}\left[\frac{1}{\infty} - \frac{1}{1}\right] = 1$

$\Rightarrow \quad \frac{p}{8} = 1 \quad \Rightarrow \quad p = 8$

4. (*b*) Let X denotes the sum of the numbers obtained when two fair dice are rolled. So, X may have values, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12.

(as 1 can't be the sum of two numbers on fair dice)

$P(X = 2) = P[\{(1, 1)\}] = \frac{1}{36}, \; P(X = 3) = P[\{(1, 2), (2, 1)\}] = \frac{2}{36}$

$P(X = 4) = P[\{(1, 3), (2, 2), (3, 1)\}] = \frac{3}{36}$

$P(X = 5) = P[\{(1, 4), (2, 3), (3, 2), (4, 1)\}] = \frac{4}{36}$

$P(X = 6) = P[\{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}] = \frac{5}{36}$

$P(X = 7) = P[\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}] = \frac{6}{36}$

$P(X = 8) = P[\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}] = \frac{5}{36}$

$P(X = 9) = P[\{(3, 6), (4, 5), (5, 4), (6, 3)\}] = \frac{4}{36}$

$P(X = 10) = P[\{(4, 6), (5, 5), (6, 4)\}] = \frac{3}{36}$

$P(X = 11) = P[\{(5, 6), (6, 5)\}] = \frac{2}{36}$

$P(X = 12) = P[\{(6, 6)\}] = \frac{1}{36}$

X	2	3	4	5	6	7	8	9	10	11	12
$P(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean $X = \Sigma X\,P(X)$

$= \frac{\left[\begin{array}{l} 2 \times 1 + 3 \times 2 + 4 \times 3 + 5 \times 4 + 6 \times 5 + 7 \times 6 \\ + 8 \times 5 + 9 \times 4 + 10 \times 3 + 11 \times 2 + 12 \times 1 \end{array}\right]}{36}$

$= \frac{252}{36} = 7$

Variance $X = \Sigma X^2 P(X) - (\text{Mean})^2$

$= \frac{\left[\begin{array}{l} 2^2 \times 1 + 3^2 \times 2 + 4^2 \times 3 + 5^2 \times 4 + 6^2 \times 5 + 7^2 \times 6 \\ + 8^2 \times 5 + 9^2 \times 4 + 10^2 \times 3 + 11^2 \times 2 + 12^2 \times 1 \end{array}\right]}{36} - 7^2$

$= \frac{1974}{36} - 49 = \frac{1974 - 1764}{36} = \frac{210}{36} = \frac{35}{6}$

Hence, $\text{SD} = \sqrt{\text{Variance}} = \sqrt{\frac{35}{6}}$

5. (a) $E(X) = \int_{-\infty}^{\infty} x f(x)dx = \int_2^4 x\left(\frac{3+2x}{18}\right)dx$

$$= \frac{1}{18}\int_2^4 (3x+2x^2)\,dx = \frac{1}{18}\left[\frac{3x^2}{2}+\frac{2x^3}{3}\right]_2^4$$

$$= \frac{1}{18}\left[\left(24+\frac{128}{3}\right)-\left(6+\frac{16}{3}\right)\right] = \frac{1}{18}\times\frac{166}{3} = \frac{83}{27}$$

6. (a) Here, total number of students = 15

The ages of students in ascending order are 14, 14, 15, 16, 16, 17, 17, 17, 18, 19, 19, 20, 20, 20, 21

Now, $P(X=14) = \frac{2}{15}$, $P(X=15) = \frac{1}{15}$,

$P(X=16) = \frac{2}{15}$, $P(X=17) = \frac{3}{15}$

$P(X=18) = \frac{1}{15}$, $P(X=19) = \frac{2}{15}$,

$P(X=20) = \frac{3}{15}$, $P(X=21) = \frac{1}{15}$

Therefore, the probability distribution of random variable X is as follows

X	14	15	16	17	18	19	20	21
Number of students	2	1	2	3	1	2	3	1
$P(X)$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

The third row gives the probability distribution of X.

Mean $X = \Sigma X\,P(X)$

$$= \frac{14\times2+15\times1+16\times2+17\times3+18\times1+19\times2+20\times3+21\times1}{15}$$

$$= \frac{28+15+32+51+18+38+60+21}{15} = \frac{263}{15} = 17.53$$

Variance $X = \Sigma X^2 P(X) - (\text{Mean})^2$

$$= \frac{\left[\begin{array}{l}(14)^2\times2+(15)^2\times1+(16)^2\times2+(17)^2\times3\\ +(18)^2\times1+(19)^2\times2+(20)^2\times3+(21)^2\times1\end{array}\right]}{15} - \left(\frac{263}{15}\right)^2$$

$$= \frac{392+225+512+867+324+722+1200+441}{15} - \left(\frac{263}{15}\right)^2$$

$$= \frac{4683}{15} - \left(\frac{263}{15}\right)^2 = 312.2 - 307.4 = 4.8$$

SD of $X = \sqrt{\text{Variance}} = \sqrt{4.8} = 2.19$

7. (a) Let the probability of success and failure be p and q, respectively.

Then, $p = 2q$ and $p + q = 1 \Rightarrow 3q = 1 \Rightarrow q = \frac{1}{3}$

$\therefore \quad p = \frac{2}{3}$

Required probability

$$= {}^6C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 + {}^6C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right) + {}^6C_6\left(\frac{2}{3}\right)^6 = \frac{496}{729}$$

8. (a) Since, $\Sigma p_i = 1$, we have

$$C + 2C + 2C + 3C + C^2 + 2C^2 + 7C^2 + C = 1$$

i.e. $\quad 10C^2 + 9C - 1 = 0$

i.e. $\quad (10C-1)(C+1) = 0$

$\Rightarrow \quad C = \frac{1}{10},\; C = -1$

Therefore, the permissible value of $C = \frac{1}{10}$

Mean $= \sum_{i=1}^{n} x_i p_i = \sum_{i=1}^{7} x_i p_i$

$$= 1\times\frac{1}{10} + 2\times\frac{2}{10} + 3\times\frac{2}{10} + 4\times\frac{3}{10} + 5\left(\frac{1}{10}\right)^2 + 6\times2\left(\frac{1}{10}\right)^2 + 7\left(7\left(\frac{1}{10}\right)^2 + \frac{1}{10}\right)$$

$$= \frac{1}{10} + \frac{4}{10} + \frac{6}{10} + \frac{12}{10} + \frac{5}{100} + \frac{12}{100} + \frac{49}{100} + \frac{7}{10}$$

$= 3.66$

9. (b) $p(0 < X < K) = 0.5$

$$\int_0^K f(x)dx = \frac{1}{2} \Rightarrow \int_0^K ae^{-ax}\,dx = \frac{1}{2}$$

$$\Rightarrow a\left[\frac{e^{-ax}}{-a}\right]_0^K = \frac{1}{2} \Rightarrow -[e^{-ax}]_0^K = \frac{1}{2}$$

$$\Rightarrow -(e^{-aK} - e^0) = \frac{1}{2} \Rightarrow -e^{-aK} + 1 = \frac{1}{2}$$

$$\Rightarrow e^{-aK} = \frac{1}{2} \Rightarrow -aK = \log\left(\frac{1}{2}\right)$$

$$\Rightarrow aK = \log 2 \Rightarrow K = \frac{1}{a}\log 2$$

10. (c) Let probability of defective bulb, $p = \frac{10}{100} = \frac{1}{10} = 0.1$

and probability of non-defective bulb, $q = 1 - 0.1 = 0.9$

Here, $\quad n = 5$

$\therefore$ P (none is defective) $= P(X=0) = {}^5C_0(0.1)^0\,(0.9)^5$

$$= 1\times(0.9)^5 = \left(\frac{9}{10}\right)^5$$

11. (a) $P(X<1) = P(0<X<1) = \int_0^1 f(x)dx = \int_0^1\left(\frac{x}{8}\right)dx$

$$= \frac{1}{8}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{8}\left[\frac{1}{2}-0\right] = \frac{1}{16}$$

$$P(X\geq2) = P(2\leq X<4) = \int_2^4 f(x)dx$$

$$= \int_2^4\left(\frac{x}{8}\right)dx = \frac{1}{8}\left[\frac{x^2}{2}\right]_2^4$$

$$= \frac{1}{16}[4^2-2^2] = \frac{12}{16} = \frac{3}{4}$$

12. (a) Let X be the money won in one throw.

Money lost in 1 throw = ₹ 1

Also, probability of getting $3 = \frac{1}{6}$

Probability of getting 1 or 6

$$\Rightarrow \quad \frac{1}{6}+\frac{1}{6}=\frac{2}{6}$$

Probability of getting any other number i.e. 2 or 4 or 5

$$=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{3}{6}$$

Then, probability distribution is

X	5	2	0
$P(X)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

Then, expectation of money that player can won

$$E(X)=\frac{5}{6}+\frac{4}{6}+0=\frac{9}{6}=₹\,1.5$$

Then, player's expected profit = ₹ 1.5 − ₹ 1 = 0.50

13. (c) Given that, $E(X)=3$ and $(E(X^2)=11$

Variance of $X=E(X^2)-[E(X)]^2=11-(3)^2=11-9=2$

14. (a) Sum of probabilities = 1

$$\Rightarrow \quad p+2p+3p+4p+5p+7p+8p+9p+10p+11p+12p=1$$

$$\Rightarrow \quad 72p=1 \Rightarrow p=\frac{1}{72}$$

15. (c) As we know, the sum of all the probability in a probability distribution is one.

$\therefore P(X=x_1)+P(X=x_2)+\ldots+P(X=x_{10})=1$

$$\Rightarrow \quad 1k+2k+3k+\ldots+10k=1$$

$$\Rightarrow \quad \frac{10\,(10+1)}{2}k=1 \Rightarrow k=\frac{1}{55}$$

16. (c) Given, distribution is

X	0	1	2	3
$P(X)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

$\therefore$ **Mean,** $m=\sum_{i=1}^{4} p_i x_i = 0\times\frac{1}{3}+1\times\frac{1}{2}+2\times 0+3\times\frac{1}{6}$

$$=0+\frac{1}{2}+0+\frac{1}{2}=1$$

Variance, $\sigma^2=\sum_{i=1}^{4} p_i (x_i-m)^2$

$$=\frac{1}{3}(0-1)^2+\frac{1}{2}(1-1)^2+0(2-1)^2+\frac{1}{6}(3-1)^2$$

$$=\frac{1}{3}+0+0+\frac{2}{3}=1$$

$$m=\sigma^2=1$$

17. (a) $P(E)=P(X=2)+P(X=3)+P(X=5)+P(X=7)$

$$=0.23+0.12+0.20+0.07=0.62$$

$$P(F)=P(X=1)+P(X=2)+P(X=3)$$

$$=0.15+0.23+0.12=0.5$$

$$P(E\cap F)=P(X=2)+P(X=3)=0.23+0.12=0.35$$

$$\therefore \quad P(E\cup F)=P(E)+P(F)-P(E\cap F)$$

$$=0.62+0.5-0.35=0.77$$

18. (a) We know that, total probability distribution is 1.

$$\therefore \quad \frac{1}{10}+k+\frac{1}{5}+2k+\frac{3}{10}+k=1$$

$$\Rightarrow \quad \frac{6}{10}+4k=1 \Rightarrow k=\frac{1}{10}$$

19. (a) We know that, sum of probability distribution is 1.

$$\therefore \quad k+2k+3k+4k=1 \Rightarrow k=\frac{1}{10}$$

Now, mean $\bar{X}=k\times1+2k\times2+3k\times3+4k\times4$

$$=k+4k+9k+16k=30k$$

$$\Rightarrow \quad \bar{X}=30\times\frac{1}{10}=3$$

20. (b) Given, $P(X=k)=\frac{(k+1)a}{3^k}$, for $x\in\{0,1,2,\ldots\infty\}$

As we know that,

$$P(0)+P(1)+P(2)+\ldots\infty=1$$

$$\Rightarrow \quad a+\frac{2a}{3}+\frac{3a}{3^2}+\ldots\infty=1 \quad \ldots(i)$$

Let $S=a\left(1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots\infty\right)$

$$\Rightarrow \quad \frac{1}{3}S=a\left(\frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^3}+\ldots\infty\right)$$

$$\therefore \quad S-\frac{1}{3}S=a\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\infty\right)$$

$$\Rightarrow \quad \frac{2}{3}S=a\left(\frac{1}{1-\frac{1}{3}}\right) \Rightarrow S=\frac{9a}{4}$$

$\therefore$ From Eq. (i) $\frac{9a}{4}=1$

$$\Rightarrow \quad a=\frac{4}{9}$$

21. (c) Let X denotes the number of aces.

Probability of selecting aces, $p=\frac{4}{52}=\frac{1}{13}$

Probability of not selecting aces, $q=1-\frac{1}{13}=\frac{12}{13}$

$$p(x=0)={}^2C_0\times\left(\frac{1}{13}\right)^0\times\left(\frac{12}{13}\right)^2=\frac{144}{169}$$

$$P(X=1)={}^2C_1\times\left(\frac{1}{13}\right)\times\left(\frac{12}{13}\right)=\frac{24}{169}$$

$$P(X=2) = {}^2C_2\left(\frac{1}{13}\right)^2 \cdot \left(\frac{12}{13}\right)^0 = \frac{1}{169}$$

$$\text{Mean} = \Sigma P_i X_i = 0 \times \frac{144}{169} + 1 \times \frac{24}{169} + 2 \times \frac{1}{169}$$

$$= \frac{24}{169} + \frac{2}{169} = \frac{2}{13}$$

22. (e) $P(X=0) = k, \quad P(X=1) = 2k\left(\frac{1}{5}\right)^1$

$$P(X=2) = 3k\left(\frac{1}{5}\right)^2, \dots$$

Since, $P(X=0) + P(X=1) + P(X=2) + \dots = 1$

$\therefore \quad k + 2k\left(\frac{1}{5}\right) + 3k\left(\frac{1}{5}\right)^2 + \dots = 1$

and $\quad \frac{k}{5} + 2k\left(\frac{1}{5}\right)^2 + \dots = \frac{1}{5}$

$$k + k\left(\frac{1}{5}\right) + k\left(\frac{1}{5}\right)^2 + \dots = \frac{4}{5}$$

$\Rightarrow \quad \dfrac{k}{1-\frac{1}{5}} = \frac{4}{5} \Rightarrow k = \frac{16}{25}$

$\therefore \quad P(X=0) = \frac{16}{25}(0+1)\left(\frac{1}{5}\right)^0 = \frac{16}{25}$

23. (c) Given, mean of Binomial variable, $np = 8$

and variance of Binomial variable, $npq = 4$

$\therefore \quad q = \frac{1}{2}$ and $p = 1 - q = 1 - \frac{1}{2} = \frac{1}{2}$

and $\quad n\left(\frac{1}{2}\right) = 8 \Rightarrow n = 16$

$\therefore P(X<3) = P(X=0) + P(X=1) + P(X=2)$

$$= {}^{16}C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{16-0} + {}^{16}C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{16-1} + {}^{16}C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{16-2}$$

$$= 1\left(\frac{1}{2}\right)^{16} + 16\left(\frac{1}{2}\right)^{16} + 120\left(\frac{1}{2}\right)^{16} = \frac{137}{2^{16}}$$

Exercise 2

1. (a) Since, the sum of all the probabilities in a probability distribution is always unity.

$\therefore \quad P(X=0) + P(X=1) + \dots + P(X=7) = 1$

$\Rightarrow \quad 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow \quad 10k^2 + 9k - 1 = 0$

$\Rightarrow \quad 10k - 1(k+1) = 0$

$\Rightarrow \quad 10k - 1 = 0 \quad [\because k \geq 0 \therefore K + 1 \neq 0]$

$\Rightarrow \quad k = \frac{1}{10}$

Now, $P(X \geq 6) = P(X=6) + P(X=7)$

$= 2k^2 + 7k^2 + k = 9k^2 + k = \frac{19}{100} \quad [\because k = 1/10]$

2. (a) $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^3 \left(\frac{x}{6} + k\right)dx = 1$

$\Rightarrow \quad \left[\frac{x^2}{12} + kx\right]_0^3 = 1 \Rightarrow \frac{3}{4} + 3k = 1$

$\Rightarrow \quad 3k = \frac{1}{4} \Rightarrow k = \frac{1}{12}$

3. (b) Given distribution is

X	1	2	3	4	5
$P(X=x)$	k	$2k$	$3k$	$2k$	k

$\therefore$ Variance $= \Sigma x_i^2 p - (\Sigma x_i p)^2$

$= (1k + 8k + 27k + 32k + 25k) - (k + 4k + 9k + 8k + 5k)^2$

$= (93k) - (27k)^2 = \left(93 \times \frac{1}{9}\right) - \left(27 \times \frac{1}{9}\right)^2 \quad \left[\because \Sigma p = 1, \text{ so } k = \frac{1}{9}\right]$

$= \frac{93}{9} - 9 = \frac{93 - 81}{9} = \frac{12}{9} = \frac{4}{3}$

4. (c) $\because f(x)$ is the pdf.

$\therefore \quad \int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^{\infty} k.e^{-\theta x}dx = 1$

$\Rightarrow \quad k\left[\frac{e^{-\theta x}}{-\theta}\right]_0^{\infty} = 1 \Rightarrow -\frac{k}{\theta}\left[\frac{1}{e^{\theta x}}\right]_0^{\infty} = 1$

$\Rightarrow \quad -\frac{k}{\theta}\left[\frac{1}{e^{\infty}} - \frac{1}{e^0}\right] = 1 \Rightarrow -\frac{k}{\theta}\left[\frac{1}{\infty} - \frac{1}{1}\right] = 1$

$\Rightarrow \quad -\frac{k}{\theta}[0-1] = 1 \Rightarrow \frac{k}{\theta} = 1$

$\therefore \quad k = \theta$

5. (a) Mean $= E(X) = \Sigma x_i \cdot P(x_i) = \frac{1}{6}(1) + \frac{1}{3}(2) + \frac{1}{3}(3) + \frac{1}{6}(4)$

$= \frac{1}{6} + \frac{2}{3} + 1 + \frac{4}{6} = \frac{1+4+6+4}{6} = \frac{15}{6} = \frac{5}{2}$

Variance $= \Sigma x_i^2 \cdot P(x_i) - [E(X)]^2$

$= \frac{1}{6}(1)^2 + \frac{1}{3}(2)^2 + \frac{1}{3}(3)^2 + \frac{1}{6}(4)^2 - \left(\frac{5}{2}\right)^2$

$= \frac{1}{6} + \frac{4}{3} + \frac{9}{3} + \frac{16}{6} - \frac{25}{4} = \frac{2 + 16 + 36 + 32 - 75}{12}$

$= \frac{86 - 75}{12} = \frac{11}{12}$

6. (*b*) $P(2 < X < 3) = \int_2^3 \left(\frac{2x+3}{18}\right) dx = \frac{1}{18}[x^2 + 3x]_2^3$

$$= \frac{1}{18}(9 + 9 - 4 - 6) = \frac{4}{9}$$

7. (*a*) Since, the given distribution is a probability distribution.

$$0 + 2p + 2p + 3p + p^2 + 2p^2 + 7p^2 + 2p = 1$$

$\Rightarrow \quad 10p^2 + 9p - 1 = 0$

$\Rightarrow \quad (10p - 1)(p + 1) = 0$

$\Rightarrow \quad p = 1/10 \qquad [\because p + 1 \neq 0]$

8. (*b*) The computation of mean and standard deviation is as follows:

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
1	0.1	0.1	0.1
2	0.2	0.4	0.8
3	0.3	0.9	2.7
4	0.4	1.6	6.4
		$\Sigma p_i x_i = 3$	$\Sigma p_i x_i^2 = 10$

$\therefore \quad$ Mean $= \Sigma p_i x_i = 3$

Variance $(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = 10 - 9 = 1$

9. (*d*) The given distribution will be a probability distribution, if

$$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1$$

$\Rightarrow \quad k + 3k + 3k + k = 1$

$\Rightarrow \quad k = \frac{1}{8}$

Computation of Variance

X	$p(x)$	$Xp(x)$	$X^2 p(x)$
0	$\frac{1}{8}$	0	0
1	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
2	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{12}{8}$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{9}{8}$
Total		$\Sigma xp(x) = \frac{12}{8}$	$\Sigma x^2 p(x) = \frac{24}{8}$

$\therefore \quad$ Variance $= \Sigma x^2 p(x) - [\Sigma xp(x)]^2$

$\Rightarrow \quad$ Variance $= \frac{24}{8} - \left(\frac{12}{8}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}$

10. (*c*) $\because f(x)$ is a pdf.

$\therefore \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \quad \int_0^3 f(x) = 1$

$\Rightarrow \quad \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx = 1$

$\Rightarrow \quad \int_0^1 (ax)dx + \int_1^2 (a)\,dx + \int_2^3 (3a - ax)dx = 1$

$\Rightarrow \quad a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + a\left[3x - \frac{x^2}{2}\right]_2^3 = 1$

$\Rightarrow \quad \frac{a}{2} + a + a\left\{\left(9 - \frac{9}{2}\right) - (6 - 2)\right\} = 1$

$\therefore \quad 2a = 1 \Rightarrow a = \frac{1}{2}$

11. (*b*) We know that, $V(X) = E(X)^2 - [E(X)]^2$

$\therefore \quad 6 = E(X)^2 - (5)^2$

$\therefore \quad E(X^2) = 25 + 6 = 31$

12. (*c*) $a = P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$

$$= 4k + 2k + k = 7k$$

$b = P(X < 3) = P(X = 0) + (P(X = 1) + P(X = 2)$

$$= k + 2k + 4k = 7k$$

$\therefore \quad a = 7k = b$

13. (*d*) $\because \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$

$\therefore \quad \int_0^4 \frac{k}{\sqrt{x}}\,dx = 1$

$\Rightarrow \quad k[2\sqrt{x}]_0^4 = 1 \Rightarrow 2k(\sqrt{4} - \sqrt{0}) = 1$

$\Rightarrow \quad 4k = 1$

$\therefore \quad k = \frac{1}{4}$

$\Rightarrow \quad P(X \geq 1) = P(1 \leq X < 4) = \int_1^4 f(x)\,dx = 2k[\sqrt{x}]_1^4$

$$= 2\left(\frac{1}{4}\right)(2 - 1) = \frac{1}{2} = 0.5$$

14. (*a*) Mean $= E(X) = \Sigma x_i \cdot P(x_i)$

$$= (0.05)(-3) + (0.45)(-1) + (0.20)\,0 + (0.25)\,1 + (0.05)\,3$$

$$= -0.15 - 0.45 + 0 + 0.25 + 0.15 = -0.2$$

15. (*b*) $E(X) = \sum x_i . P(x_i) = \frac{1}{8}(1) + \frac{1}{2}(2) + \frac{1}{8}(3) + \frac{1}{4}(4)$

$$= \frac{1}{8} + 1 + \frac{3}{8} + 1 = \frac{5}{2}$$

Now, $\quad V(X) = E(X^2) - [E(X)]^2$

$$= \sum x_i^2 . P(x_i) - \left(\frac{5}{2}\right)^2$$

$$= \frac{1}{8}(1)^2 + \frac{1}{2}(2)^2 + \frac{1}{8}(3)^2 + \frac{1}{4}(4)^2 - \frac{25}{4}$$

$$= \frac{1}{8} + 2 + \frac{9}{8} + 4 - \frac{25}{4}$$

$$= \frac{1 + 16 + 9 + 32 - 50}{8}$$

$$= \frac{58 - 50}{8} = \frac{8}{8} = 1$$

16. (*c*) $F(x) = \int_0^x f(t)\,dt = \int_0^x 3(1-2t^2)\,dt = [3t - 2t^3]_0^x = 3x - 2x^3$

17. (*a*) The given distribution is a probability distribution.

$\therefore \quad P(X=-2) + P(X=-1) + P(X=0) + P(X=1) + P(X=2) + P(X=3) = 1$

$\Rightarrow \quad \frac{1}{10} + k + \frac{1}{5} + 2k + \frac{3}{10} + k = 1$

$\Rightarrow \quad 4k = \frac{4}{10} \quad \Rightarrow \quad k = \frac{1}{10}$

18. (*d*) The given distribution is the probability distribution.

$\therefore \quad \sum_{r=0}^{8} P(X=r) = 1$

$\Rightarrow \quad a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$\Rightarrow \quad a = \frac{1}{81}$

19. (*d*) Since, $\frac{1+3p}{4}, \frac{1-p}{4}, \frac{1+2p}{4}$ and $\frac{1-4p}{4}$ are probabilities when X takes value $-1, 0, 1$ and 2 respectively. Therefore, each is greater than or equal to 0 and less than or equal to 1.

i.e. $\quad 0 \le \frac{1+3p}{4} \le 1,$

$0 \le \frac{1-p}{4} \le 1, \; 0 \le \frac{1+2p}{4} \le 1$

and $\quad 0 \le \frac{1-4p}{4} \le \frac{1}{4}$

$\Rightarrow \quad -\frac{1}{3} \le p \le \frac{1}{4}$

Let $\overline{X}$ be the mean of X. Then,

$\overline{X} = -1 \times \frac{1+3p}{4} + 0 \times \frac{1-p}{4} + 1 \times \frac{1+2p}{4} + 2 \times \frac{1-4p}{4}$

$\Rightarrow \quad \overline{X} = \frac{2-9p}{4}$

Now, $\quad -\frac{1}{3} \le p \le \frac{1}{4}$

$\Rightarrow \quad 3 \ge -9p \ge -\frac{9}{4}$

$\Rightarrow \quad -\frac{1}{4} \le 2 - 9p \le 5$

$\Rightarrow \quad -\frac{1}{16} < \frac{2-9p}{4} \le \frac{5}{4}$

$\Rightarrow \quad -\frac{1}{16} \le X \le \frac{5}{4}$

20. (*d*) Given, mean, $np = 4$ ...(i)

and variance, $npq = 2$ ...(ii)

On dividing Eq. (ii) by Eq. (i), we get

$\frac{npq}{np} = \frac{2}{4} \quad \Rightarrow \quad q = \frac{1}{2}$

$\therefore \quad p = \frac{1}{2}$

From Eq. (i), $\quad n = \frac{4}{1/2} = 8$

Now, $\quad P(X=1) = {}^nC_1 p^1 q^{n-1}$

$= {}^8C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^7 = \frac{8!}{1!\,7!} \times \frac{1}{2^8} = \frac{8}{2^8} = \frac{1}{32}$

21. (*b*) $E(X) = 3 \times \frac{1}{3} + 4 \times \frac{1}{4} + 12 \times \frac{5}{12} = 7$

22. (*c*) Here, $p = \frac{1}{3}, q = \frac{2}{3}$

Probability of guessing a 4 or more correct answer

$= {}^5C_4 \left(\frac{1}{3}\right)^4 \cdot \frac{2}{3} + {}^5C_5 \left(\frac{1}{3}\right)^5$

$= 5 \cdot \frac{2}{3^5} + \frac{1}{3^5} = \frac{11}{3^5}$